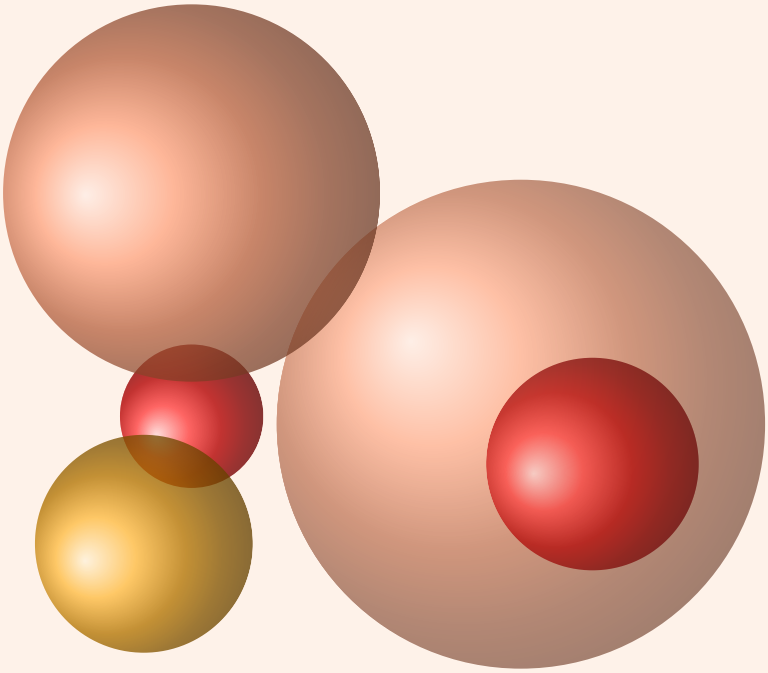
Pythagorean theorem 1.0

AlterMundus



Alain Matthes

January 26, 2011

http://altermundus.fr http://altermundus.com

Pythagorean theorem

AlterMundus

Alain Matthes

This document presents some pythagoras theorem proof. I made all the pictures with my package ***tkz-euclide.sty V1.13 c****.*

☞ Firstly, I would like to thank **Till Tantau** for the beautiful LATEX package, namely TikZ.

☞ I am grateful to **Michel Bovani** for providing the **fourier** font.

Contents

1 The Pythagoras' theorem **4**
1.1 The picture . . . 4
1.2 Interpretation . . . 4
1.3 Code of the picture . . . 5

2 Modern proof by algebra **6**

3 Proof using similar triangles **8**
3.1 The picture . . . 8
3.2 proof . . . 9
3.3 The code . . . 9

4 Proof by rearrangement **10**

5 Proof using area subtraction **11**
5.1 A picture and a proof . . . 11

6 Proof using area comparaison **12**
6.1 A picture, a proof and the code . . . 12

7 Sum of two squares **13**
7.1 Picture : Sum of two squares . . . 13
7.2 Proof with rearrangements . . . 13
7.3 Codes of pictures . . . 14

8 Proof with InCircle **16**
8.1 The picture . . . 16
8.2 Proof . . . 17
8.3 Code of the picture . . . 18

9 Proof with a trapezoid **19**
9.1 The picture . . . 19
9.2 Proof . . . 19
9.3 Code of the picture . . . 20

SECTION 1

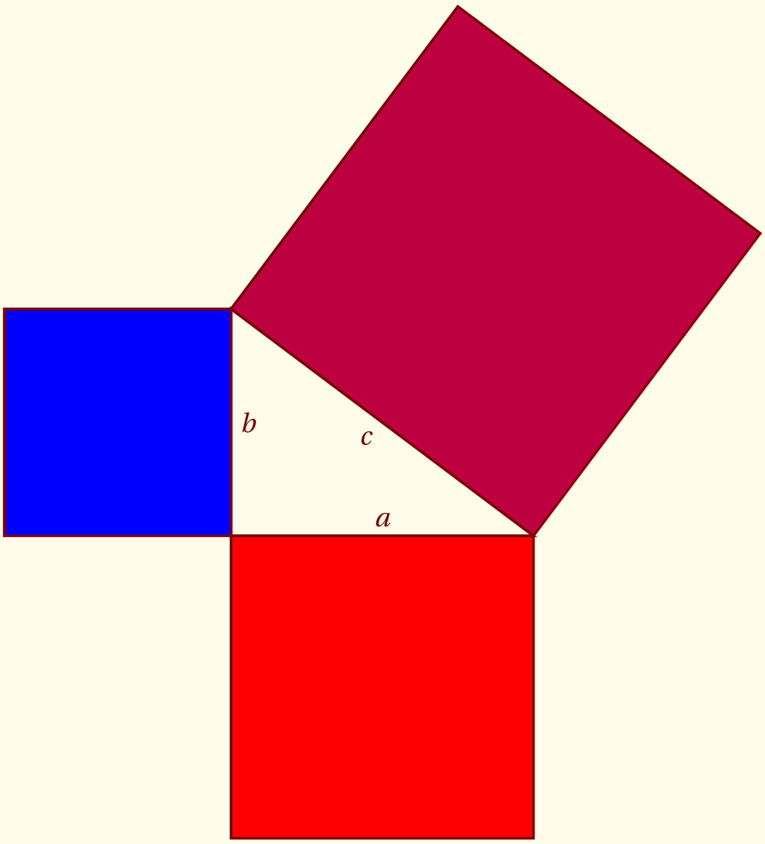
The Pythagoras' theorem

The most well-known theorem in Euclid Geometry is about the right-angle triangle, commonly attributed to Pythagoras but Greek, Chinese and Babylonian mathematicians have known about this for a long time. The theorem is of fundamental importance in Euclidean Geometry where it serves as a basis for the definition of distance between two points.

The Pythagoras' theorem

In any right angled triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides other than the hypotenuse).

1.1 The picture



1.2 Interpretation

In this picture, the area of the blue square added to the area of the red square makes the area of the purple square.

This is usually summarised as:

The square on the hypotenuse is equal to the sum of the squares on the other two sides.

If we let c be the length of the hypotenuse and a and b be the lengths of the other two sides, the theorem can be expressed as the equation

$$a^2 + b^2 = c^2$$

or, solved for c:

$$c = \sqrt{a^2 + b^2}.$$

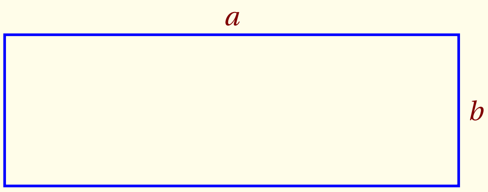
1.3 Code of the picture

```
\begin{tikzpicture}
   \tkzDefPoint(0,0){C}
   \tkzDefPoint(4,0){A}
   \tkzDefPoint(0,3){B}
   \tkzDefSquare(B,A)      \tkzGetPoints{E}{F}
   \tkzDefSquare(A,C)      \tkzGetPoints{G}{H}
   \tkzDefSquare(C,B)       \tkzGetPoints{I}{J}
   \tkzFillPolygon[color  =  red   ](A,C,G,H)
   \tkzFillPolygon[color  =  blue  ](C,B,I,J)
   \tkzFillPolygon[color  =  purple](B,A,E,F)
   \tkzDrawPolygon[line width  =  1pt](A,B,C)
   \tkzDrawPolygon[line width  =  1pt](A,C,G,H)
   \tkzDrawPolygon[line width  =  1pt](C,B,I,J)
   \tkzDrawPolygon[line width  =  1pt](B,A,E,F)
   \tkzLabelSegment[above](C,A){$a$}
   \tkzLabelSegment[right](B,C){$b$}
   \tkzLabelSegment[below left](B,A){$c$}
\end{tikzpicture}
```

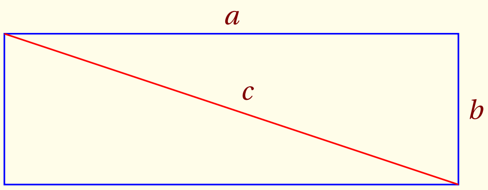
SECTION 2

Modern proof by algebra

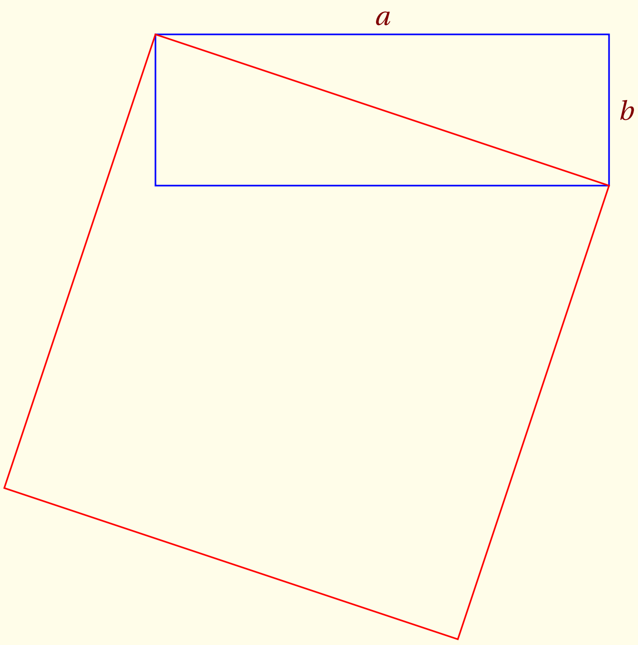
Draw a rectangle with two sides of a and b (say the lower left).



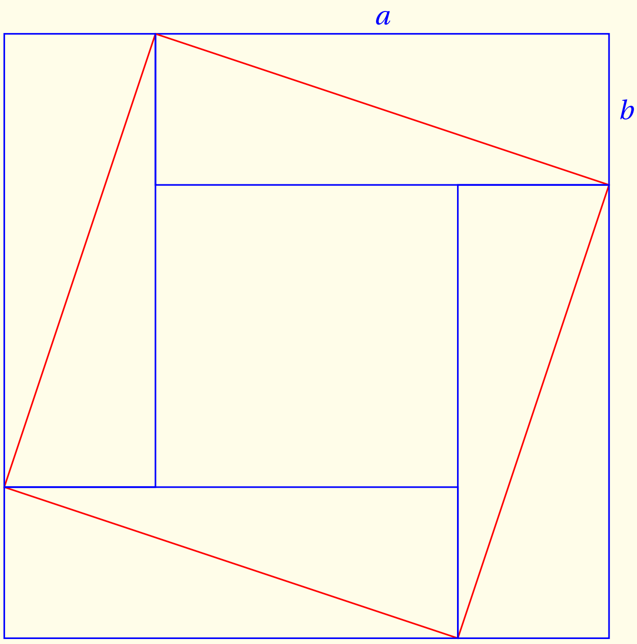
Draw a diagonal of this rectangle (call it c).



Draw a square using the diagonal as its side (shown in red).



Replicate the original rectangle three times around the red square.



We have a square of side $a+b$, inscribed in it is a red square of side c, and four copies of the original right triangle.

$$\begin{aligned} 4\frac{ab}{2} + c^2 &= (a+b)^2 \\ 2ab + c^2 &= a^2 + 2ab + b^2 \\ c^2 &= a^2 + b^2 \end{aligned}$$

Another possibilty

Area of red square = Areas of 4 triangles + Area of small square.

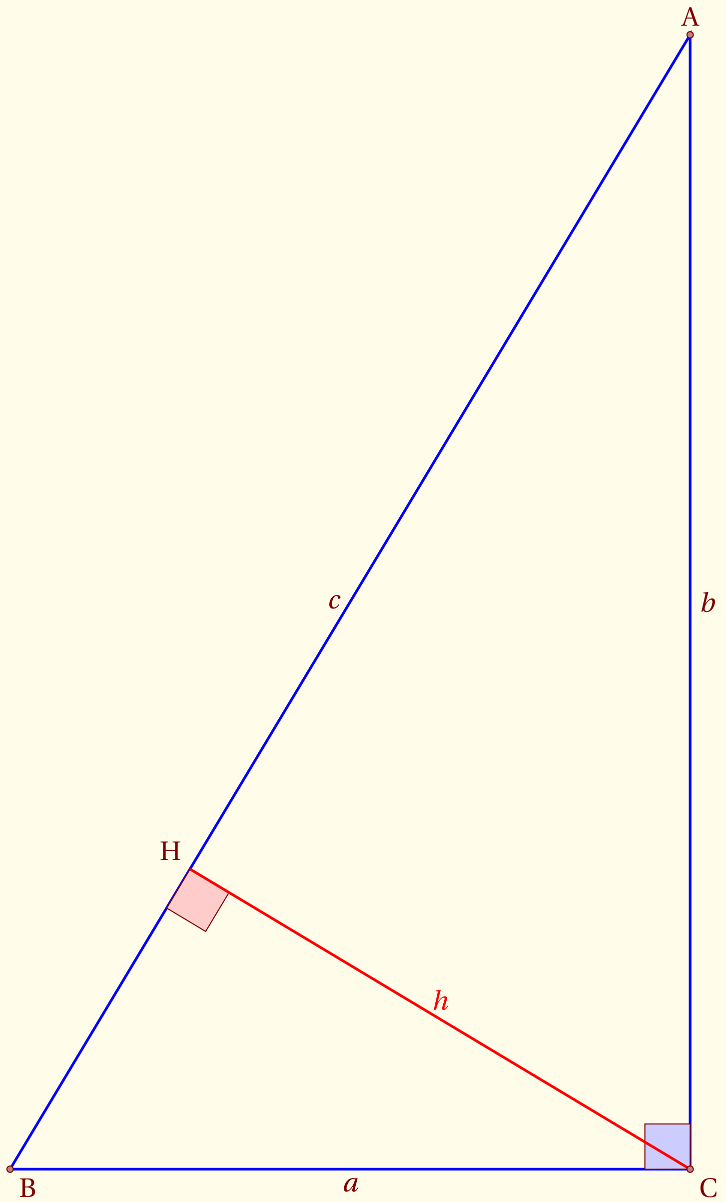
The algebra here asserts that

$$\begin{aligned} 4\frac{ab}{2} + (b-a)^2 &= c^2 \\ 2ab + b^2 - 2ab + a^2 &= c^2 \\ a^2 + b^2 &= c^2 \end{aligned}$$

SECTION 3

Proof using similar triangles

3.1 The picture



3.2 proof

Like many of the proofs of the Pythagoras' Theorem, this one is based on the proportionality of the sides of two similar triangles.
Let ABC represents a right triangle, with the right angle located at C, as shown on the figure.
We draw the altitude from point C, and call H its intersection with the side AB. The new triangle ACH is similar to the triangle ABC, because they both have a right angle (by definition of the altitude), and they share the angle at A. The triangle BCH is also similar to ABC. The two similarities lead to the two ratios:

Proportionality of the sides of two similar triangles

$$\frac{\mathrm{BH}}{a} = \frac{a}{c} \text{ and } \frac{\mathrm{HA}}{b} = \frac{b}{c}$$

These can be written as

$$a^2 = c \times \mathrm{BH} \text{ and } b^2 = c \times \mathrm{HA}.$$

Summing these two equalities, we obtain

$$a^2 + b^2 = c \times \mathrm{BH} + c \times \mathrm{AH} = c \times (\mathrm{BH} + \mathrm{HA}) = c^2.$$

In other words, the Pythagoras' Theorem:

$$a^2 + b^2 = c^2$$

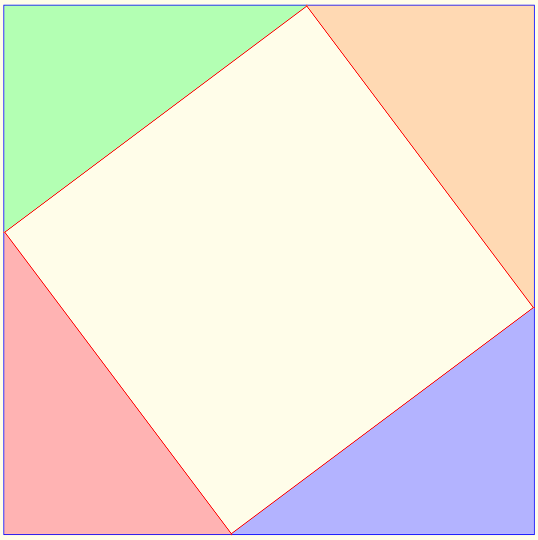
3.3 The code

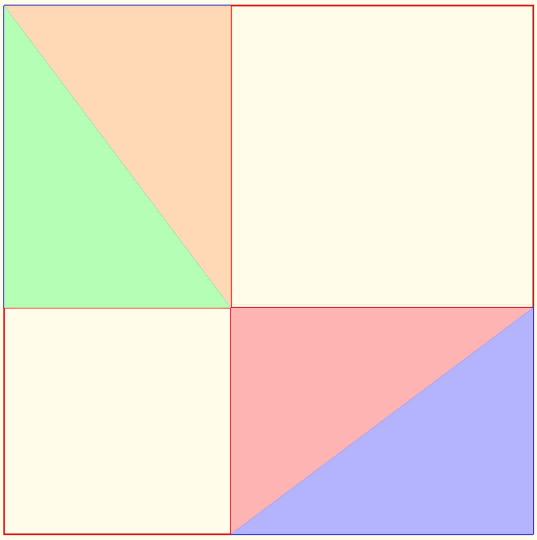
```
\begin{tikzpicture}
  \tkzInit[ymax=6]\tkzClip[space=1]
  \tkzDefPoint(10,6){C}
  \tkzDefPoint( 0,6){A}
  \tkzDefPoint(10,0){B}
  \tkzSetUpLine[color=blue,line width=1pt]
  \tkzLabelSegment[right](C,B){$c$}
  \tkzLabelSegment(A,C){$b$}
  \tkzLabelSegment(A,B){$a$}
  \tkzDrawPolygon(A,B,C)
  \tkzDefPointBy[projection = onto B--A](C) \tkzGetPoint{H}
  \tkzMarkRightAngle[size=.4,fill=blue!20](B,C,A)
  \tkzMarkRightAngle[size=.4,fill=red!20](C,H,A)
  \tkzDrawSegment[color=red](C,H)
  \tkzLabelSegment[color=red](C,H){$h$}
  \tkzDrawPoints(A,B,C)
  \tkzLabelPoints(B) \tkzLabelPoints[above](A,C)
\end{tikzpicture}
```

SECTION 4

Proof by rearrangement

The area of each large square is $(a+b)^2$. In both figures, the area of four identical triangles is removed. The remaining areas in white, a^2+b^2 and c^2, are equal.





```
\begin{tikzpicture}
 \tkzDefPoint(0,0){A} \tkzDefPoint(7,0){B}
 \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
 \tkzDrawPolygon[color  =  blue](B,C,D,A)
 \tkzDefPoint(3,0){I} \tkzDefPoint(7,3){J}
 \tkzDefSquare(I,J)   \tkzGetPoints{K}{L}
 \tkzDrawPolygon[color=red](I,J,K,L)  \tkzFillPolygon[color=red!30](A,I,L)
 \tkzFillPolygon[color=blue!30](I,B,J)\tkzFillPolygon[color=green!30](K,D,L)
 \tkzFillPolygon[color=orange!30](K,C,J)

\end{tikzpicture}
```

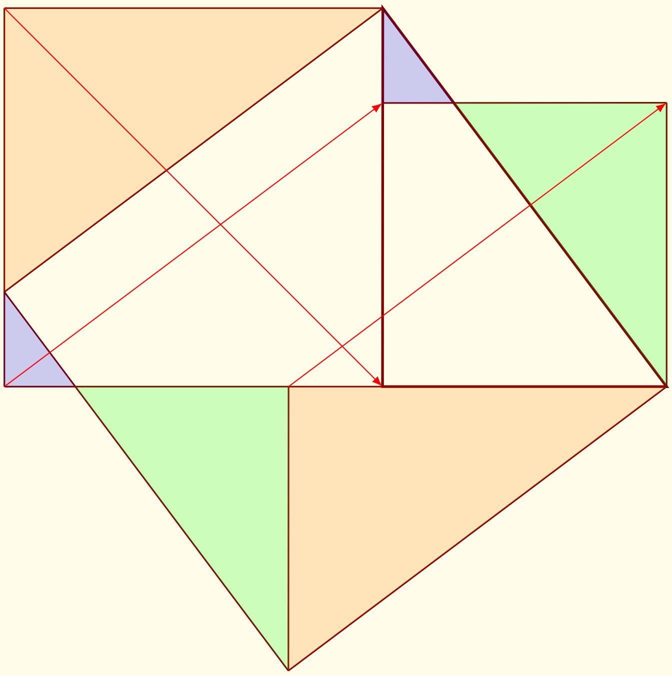
```
\begin{tikzpicture}
 \tkzDefPoint(0,0){A} \tkzDefPoint(7,0){B}
 \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
 \tkzDefPoint(3,0){I} \tkzDefPoint(7,3){J}
 \tkzDefSquare(A,I)  \tkzGetPoints{M}{L}
 \tkzDefSquare(M,J)  \tkzGetPoints{C}{P}
 \tkzDrawPolygon[color=red](A,I,M,L) \tkzDrawPolygon[color=red](M,J,C,P)
 \tkzDrawSegments[color=blue](I,J D,M I,B B,J P,D D,L)
 \tkzFillPolygon[color=red!30](M,J,I)  \tkzFillPolygon[color=green!30](D,M,L)
 \tkzFillPolygon[color=blue!30](I,B,J) \tkzFillPolygon[color=orange!30](D,P,M)
\end{tikzpicture}
```

SECTION 5

Proof using area subtraction

5.1 A picture and a proof

The red arrows show three translations of triangles.



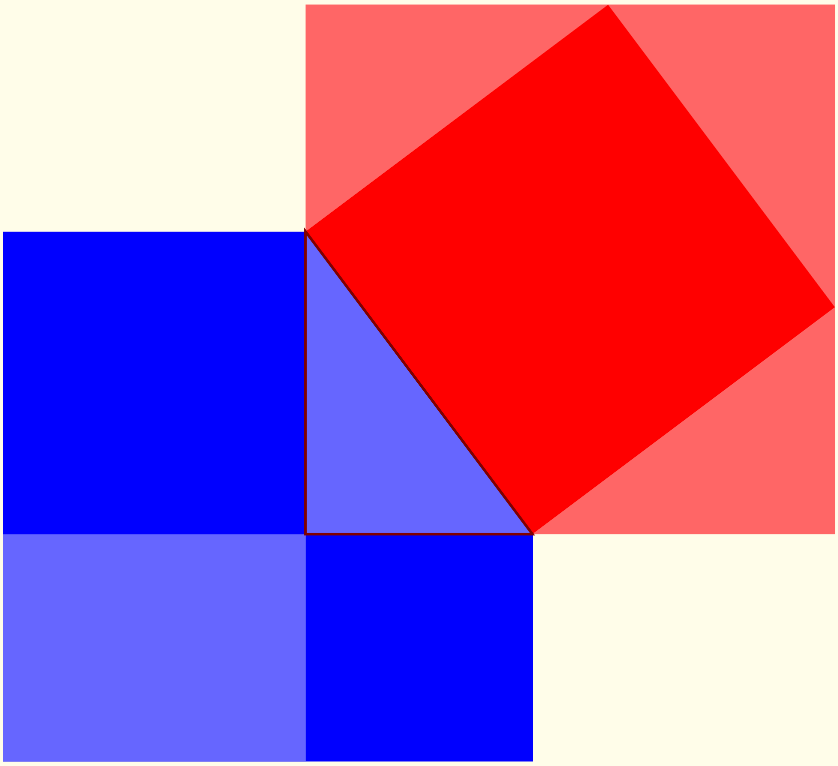
```
\begin{tikzpicture}
   \tkzInit
   \tkzDefPoint(0,0){A} \tkzDefPoint(4,0){B}
   \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
   \tkzDefPoint(4,3){I}
   \tkzDefSquare(I,B) \tkzGetPoints{J}{K}
   \tkzDefSquare(J,C)  \tkzGetPoints{L}{M}
   \tkzDrawPolygon[lw = 1pt](B,C,J)%
   \tkzDefPointBy[projection = onto A--B ](M) \tkzGetPoint{T}
   \tkzDrawSegments[](K,J I,K L,M M,J C,L C,D D,A A,B M,T)
   \tkzDefInterLL(A,B)(L,M)\tkzGetPoint{R}
   \tkzDefInterLL(I,K)(C,J)\tkzGetPoint{S}
   \tkzFillPolygon[color = blue,opacity = .2](S,C,I)
   \tkzFillPolygon[color = blue,opacity = .2](R,L,A)
   \tkzFillPolygon[color = orange,opacity = .2](D,C,L)
   \tkzFillPolygon[color = orange,opacity = .2](T,J,M)
   \tkzFillPolygon[color = green,opacity = .2](J,K,S)
   \tkzFillPolygon[color = green,opacity = .2](T,R,M)
   \tkzDrawVector[color = red](D,B A,I T,K)
\end{tikzpicture}
```

SECTION 6

Proof using area comparaison

6.1 A picture, a proof and the code

This is a variant of the above proof. Two regions - the red and the blue - are equal.



```
\begin{tikzpicture}[scale=1]
 \tkzDefPoint(0,0){A} \tkzDefPoint(3,0){B} \tkzDefPoint(0,4){C}
 \tkzDefSquare(B,A)   \tkzGetPoints{D}{E}
 \tkzDefSquare(A,C)   \tkzGetPoints{F}{G}
 \tkzDefSquare(C,B)   \tkzGetPoints{H}{I}
 \path[fill = blue,opacity = 1] (E) rectangle (F);
 \path[fill = blue!60,opacity = 1] (D)-|(G)--(A)--cycle;
 \path[fill = blue!60] (A)--(B)--(C)--cycle;
 \path[fill = red!60] (B)-|(H)--(B)--cycle;
 \path[fill = red!60] (H)|-(I)--cycle;
 \path[fill = red!60] (I)-|(C)--cycle;
 \tkzFillPolygon[color = blue](B,A,D,E) \tkzFillPolygon[color = blue](A,C,F,G)
 \tkzFillPolygon[color = red,opacity = 1](C,B,H,I)
 \tkzDrawPolygon[line width = 1pt](A,B,C)
\end{tikzpicture}
```

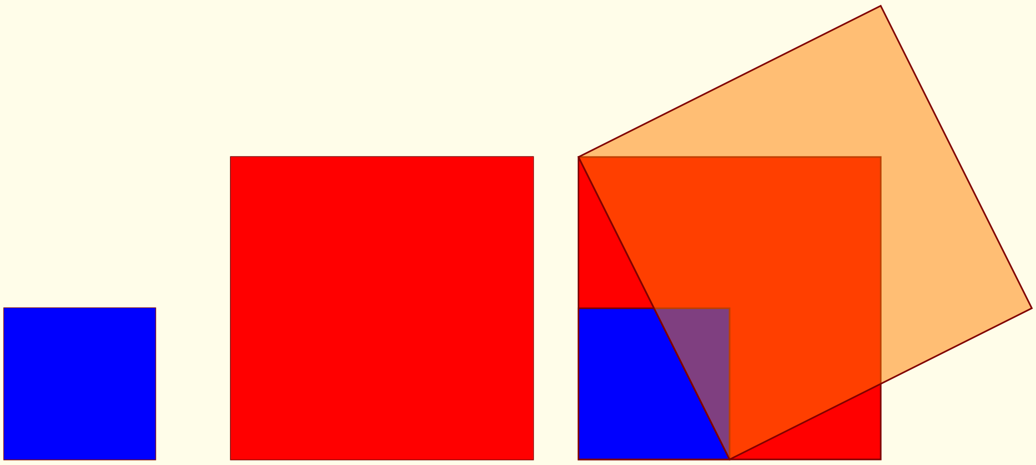
SECTION 7

Sum of two squares

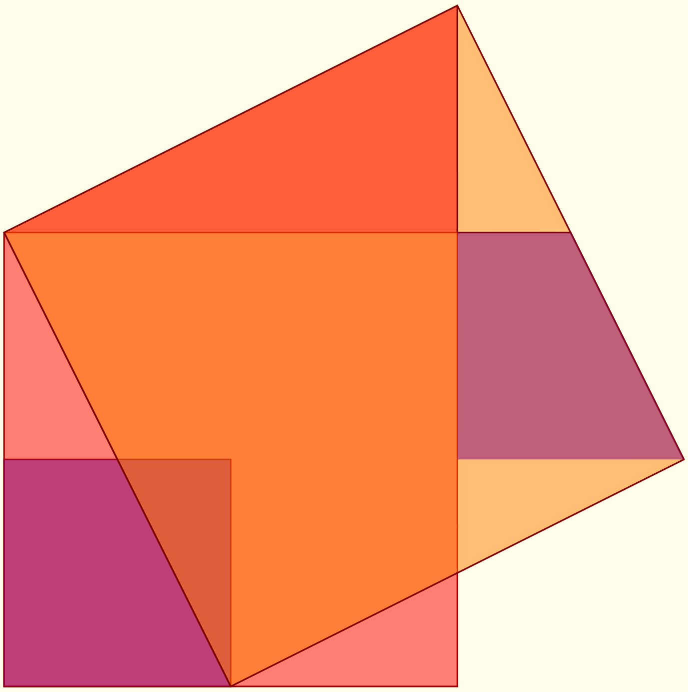
7.1 Picture : Sum of two squares

The aera of the orange square is $a^2 + b^2$

We start with two squares with sides a and b, respectively, placed side by side. The total area of the two squares is $a^2 + b^2$.



7.2 Proof with rearrangements



7.3 Codes of pictures

```
\begin{tikzpicture}[scale = 2]
  \tkzDefPoint(0,0){A} \tkzDefPoint(0,1){B}
  \tkzDefSquare(A,B) \tkzGetPoints{C}{D}
  \tkzDefPoint(2,0){I}\tkzDefPoint(4,0){J}
  \tkzDefSquare(I,J) \tkzGetPoints{K}{L}
  \tkzDrawPolygon(A,B,C,D)
  \tkzDrawPolygon(I,J,K,L)
  \tkzFillPolygon[color = blue](A,B,C,D)
  \tkzFillPolygon[color = red](I,J,K,L)
\end{tikzpicture}
```

The aera of the orange square is $a^2 + b^2$

```
  \begin{tikzpicture}[scale = 2]
     \tkzDefPoint(0,0){I'}\tkzDefPoint(2,0){J'}
     \tkzDefPoint(1,0){B'}
     \tkzDefSquare(I',J') \tkzGetPoints{K'}{L'}
     \tkzFillPolygon[color = red](I',J',K',L')
     \tkzDrawPolygon(I',J',K',L')
     \tkzDefSquare(I',B')  \tkzGetPoints{C'}{D'}
     \tkzFillPolygon[color = blue](I',B',C',D')
     \tkzDrawSegment(L',B')
     \tkzDefSquare(L',B')  \tkzGetPoints{M}{N}
     \tkzDrawPolygon(I',B',C',D')
     \tkzFillPolygon[color   = orange,opacity = .5](L',B',M,N)%
     \tkzDrawPolygon(L',B',M,N)
  \end{tikzpicture}
```

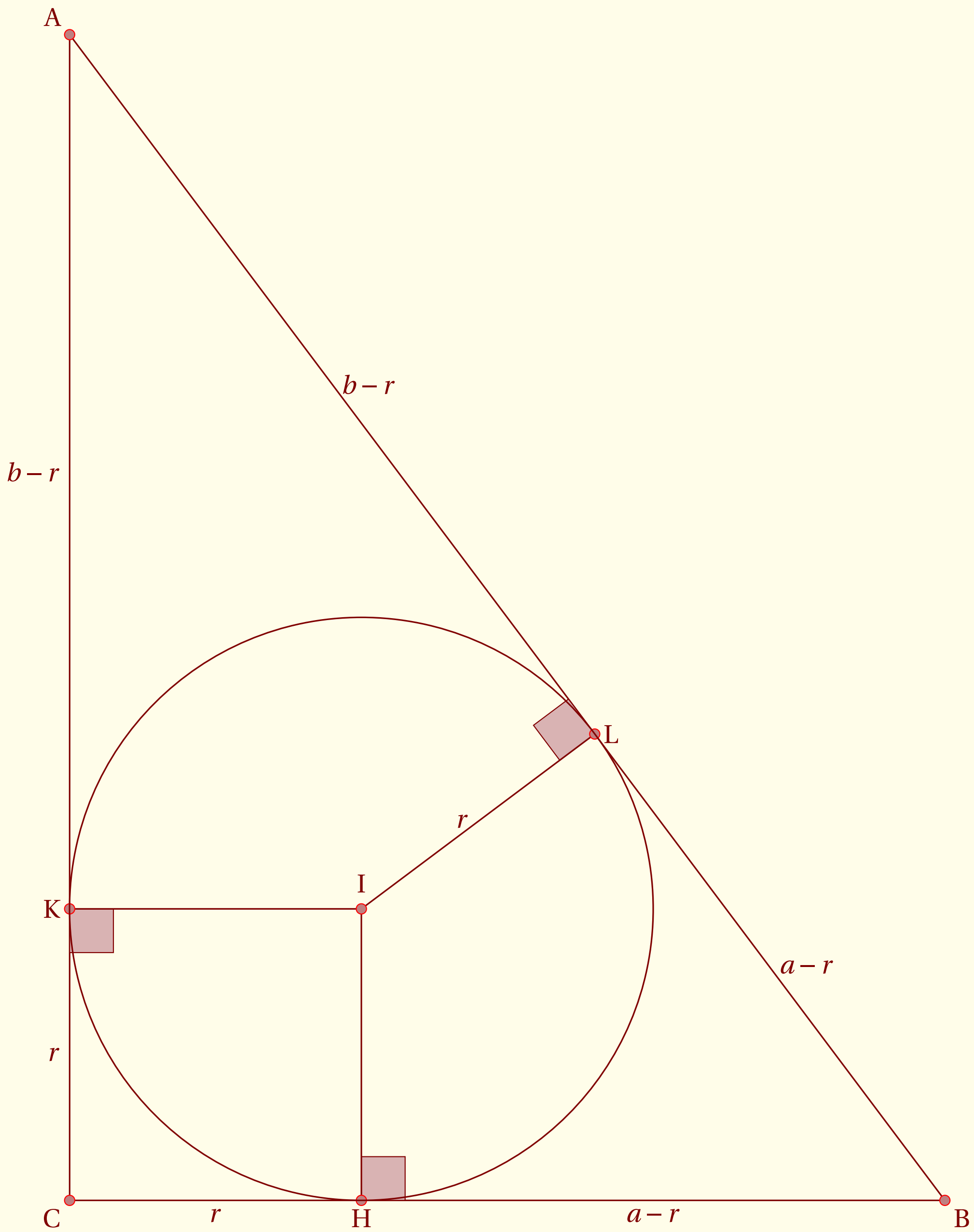
```
  \begin{tikzpicture}[scale = 2]
   \tkzDefPoint(0,0){I'}  \tkzDefPoint(2,0){J'}
   \tkzDefPoint(1,0){B'}
   \tkzDefSquare(I',J')    \tkzGetPoints{K'}{L'}
   \tkzFillPolygon[color = red](I',J',K',L')
   \tkzDrawPolygon(I',J',K',L')
   \tkzDefSquare(I',B')    \tkzGetPoints{C'}{D'}
   \tkzFillPolygon[color = blue](I',B',C',D')
   \tkzDrawSegment(L',B')
   \tkzDefSquare(L',B')    \tkzGetPoints{M}{N}
   \tkzDrawPolygon(I',B',C',D')
   \tkzFillPolygon[color   = orange, opacity = .5](L',B',M,N)
   \tkzDrawPolygon(L',B',M,N)
\end{tikzpicture}
```

```
\begin{tikzpicture}[scale = 2]
  \tkzDefPoint(0,0){I'}%
  \tkzDefPoint(2,0){J'}%
  \tkzDefPoint(1,0){B'}
  \tkzDefSquare(I',J') \tkzGetPoints{K'}{L'}
  \tkzDefSquare(I',B') \tkzGetPoints{C'}{D'}
  \tkzDefSquare(L',B') \tkzGetPoints{M}{N}
  \tkzDrawPolygon(I',B',C',D')
  \tkzDrawPolygon(I',J',K',L')
  \tkzFillPolygon[color   = blue, opacity = .5](I',B',C',D')
  \tkzFillPolygon[color   = red,opacity = .5](I',J',K',L')
  \tkzFillPolygon[color   = orange,opacity = .5](L',B',M,N)
  \tkzFillPolygon[color   = red,opacity = .5](K',N,L')
  \tkzDrawPolygon(L',B',M,N)%
  \tkzInterLL(L',K')(M,N)    \tkzGetPoint{S}
  \tkzInterLL(D',C')(J',K')  \tkzGetPoint{T}
  \tkzInterLL(B',M)(J',K')   \tkzGetPoint{U}
  \tkzDrawSegments(L',B' K',N S,M K',S)
  \tkzFillPolygon[color   = blue!50!red,opacity = .5](T,M,S,K')
\end{tikzpicture}
```

SECTION 8

Proof with InCircle

8.1 The picture



8.2 Proof

Let ABC represents a right triangle, with the right angle located at C, as shown on the figure. Let a, b and c the lengths of the three sides; c is the length of the hypothenuse.
Let r and s be the radius of the incercle and the semiperimeter of the triangle.
a, b and c can be regarded in relation to r and they may be expressed with r: $a = r + (a - r)$, $b = r + (b - r)$ and $c = (a - r) + (b - r)$.
In a right triangle, we have the relation $r = s - c$. From the diagram, the hypotenuse AB is split in two pieces: $(a - r)$ and $(b - r)$, the length of the hypothenuse is $c = (a - r) + (b - r)$.
The perimeter is a function of r

$$2s = a + b + c = r + (a - r) + r + (b - r) + (a - r) + (b - r) = 2a + 2b - 2r$$

so we can expressed r with s and c

$$2r = a + b - c = 2s - 2c \text{ and } r = s - c.$$

Compute the area of the triangle ABC in three different ways.

1. ABC is a right triangle so,
$$\text{Area(ABC)} = \frac{1}{2}ab.$$
2. The aera of a triangle is given by rs with $s = \frac{1}{2}(a + b + c)$.
$$\text{Area(ABC)} = \text{Area(AIC)} + \text{Area(CIB)} + \text{Area(BIA)}$$
The aera of triangle AIC is given by $\frac{1}{2} \times \text{AC} \times \text{IK} = \frac{1}{2}br$, therefore and by analogy :
$$\text{Area(ABC)} = \frac{1}{2}r(a + b + c) = rs$$
3. In a right triangle, we have the relation $r = s - c$ (see above) so,
$$\text{Area(ABC)} = rs = s(s - c)$$
$$rs = (s - c)s = \left(\frac{a + b + c}{2} - c\right)\left(\frac{a + b + c}{2}\right) = \frac{(a + b - c)(a + b + c)}{4} = \frac{1}{2}ab$$
Simplifications yield
$$(a + b)^2 - c^2 = 2ab$$
and
$$a^2 + b^2 = c^2$$

8.3 Code of the picture

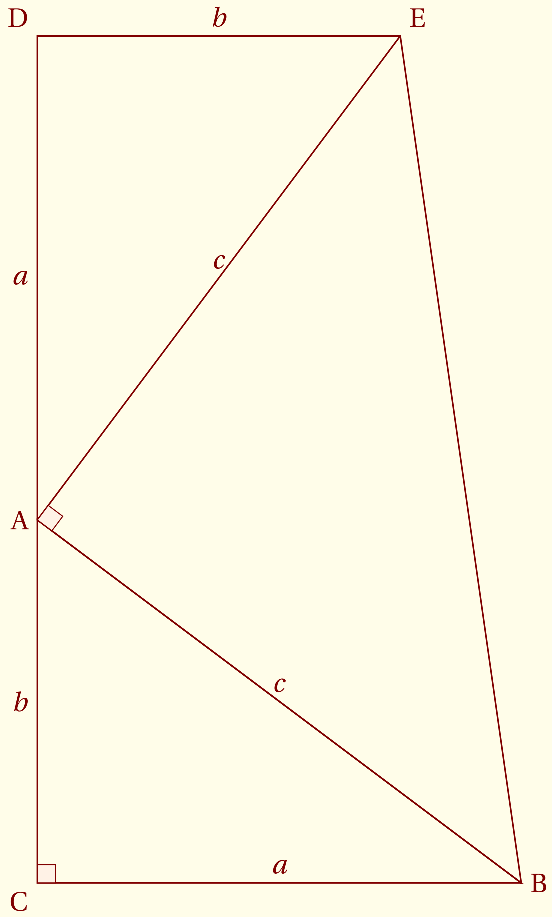
```
\begin{tikzpicture}[scale=1.5]
  \tkzInit[xmax=6,ymax=8]
  \tkzClip[space = 1]
  \tkzDefPoint(0,8){A}
  \tkzDefPoint(6,0){B}
  \tkzDefPoint(0,0){C}
  \tkzDrawPolygon(A,B,C)
  \tkzDefCircle[in](A,B,C)\tkzGetPoint{I}
  \tkzDefPointBy[projection = onto B--C](I)\tkzGetPoint{H}
  \tkzDefPointBy[projection = onto A--C](I)\tkzGetPoint{K}
  \tkzDefPointBy[projection = onto A--B](I)\tkzGetPoint{L}
  \tkzMarkRightAngles[size=.3,fill=Maroon!30](A,L,I B,H,I C,K,I)
  \tkzDrawSegments(I,L I,H I,K)
  \tkzDrawPoints[size=10,color=red](A,B,C,I,H,L,K)
  \tkzDrawCircle(I,H)
  \tkzLabelPoint[below left](C){$C$}
  \tkzLabelPoint[below right](B){$B$}
  \tkzLabelPoint[above left](A){$A$}
  \tkzLabelPoint[below](H){$H$}
  \tkzLabelPoint[left](K){$K$}
  \tkzLabelPoint[right](L){$L$}
  \tkzLabelPoint[above=3pt](I){$I$}
  \tkzLabelSegment[below](C,H){$r$}
  \tkzLabelSegment[below](H,B){$a-r$}
  \tkzLabelSegment[left](C,K){$r$}
  \tkzLabelSegment[right](L,B){$a-r$}
  \tkzLabelSegment[right](A,L){$b-r$}
  \tkzLabelSegment[left](I,L){$r$}
  \tkzLabelSegment[left](A,K){$b-r$}
 \end{tikzpicture}
```

SECTION 9

Proof with a trapezoid

9.1 The picture

This proof, discovered by President J.A. Garfield in 1876



9.2 Proof

The formula for the area of a trapezoid is the key, but this aera can be computed as the sum of areas of the three triangles ABC, ADE and EAB (EAB is an isosceles right triangle)

$$\frac{1}{2}(a+b)(a+b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$

Simplifications yield $a^2 + b^2 = c^2$.

9.3 Code of the picture

```
\begin{tikzpicture}[scale=.5]
  \tkzInit
  \tkzDefPoint(0,0){C}
  \tkzDefPoint(8,0){B}
  \tkzDefPoint(0,6){A}
  \tkzDefPoint(0,14){D}
  \tkzDefPoint(6,14){E}
  \tkzLabelPoints[below left](C)
  \tkzLabelPoints[right](B)
  \tkzLabelPoints[left](A)
  \tkzLabelPoints[above left](D)
  \tkzLabelPoints[above right](E)
  \tkzDrawPolygon(A,B,C)
  \tkzDrawPolygon(A,D,E,B)
  \tkzDrawSegment(E,A)
  \tkzMarkRightAngles(A,C,B B,A,E)
  \tkzLabelSegment(A,B){$c$}
  \tkzLabelSegment(A,C){$b$}
  \tkzLabelSegment(B,C){$a$}
  \tkzLabelSegment(D,E){$b$}
  \tkzLabelSegment(D,A){$a$}
  \tkzLabelSegment(A,E){$c$}
\end{tikzpicture}
```